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An Extension of the Corollary to Steinmetz's Theorem

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In two recent papers, "A Simpler Proof of a Theorem of Steinmetz" (1989, *J. Math. Anal. Appl.* **143**, 290–294) and "An Extension of a Theorem of Steinmetz" (1991, *J. Math. Anal. Appl.* **156**, 287–292), the present authors gave a new proof of Steinmetz's theorem that avoided use of the Second Fundamental Theorem of Nevanlinna. It was also shown in the second paper how to generalize Steinmetz's theorem to handle the case where the inner function is not precisely the same in all cases. In this paper we shall extend these results even further.

I. MAIN RESULTS

In [4] Steinmetz proves the following:

THEOREM A. *Given a functional equation of the form*

$$\sum_j F_j(g)h_j = 0, \tag{1}$$

where each $F_j(z)$ and each $h_j(z)$ are meromorphic in the complex plane, where g is entire, and where each $T(r, h_j) = O(T(r, g))$, there exist polynomials $p_j(z)$, not all zero, such that

$$\sum_j p_j(g)h_j = 0. \tag{2}$$

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The corollary to Steinmetz's theorem says that under the conditions of the theorem, there exist polynomials $q_j(z)$, not all zero, such that

$$\sum_j F_j(z)q_j(z) = 0.$$

In the authors' generalization of Steinmetz's theorem in [2], each $F_j(g)$ is replaced by $F_j(g\Psi_j)$, where each Ψ_j is a function having a small Nevanlinna characteristic function compared with the Nevanlinna characteristic function of g. Thus, Eq. (1) above in the hypothesis is replaced by

$$\sum_j F_j(\Psi_j g)h_j = 0. \tag{1'}$$

Also, it was assumed that

$$F_1(0) \neq 0.$$

In our conclusion, Eq. (2) is replaced by

$$\sum_j p_j(g, \Psi_1, \Psi_2, ..., \Psi_n)h_j(z) = 0, \tag{2'}$$

where the p_j are polynomials in the variables $\Psi_1, ..., \Psi_n$, and g. For technical reasons, having to do with the scope of our generalization of Steinmetz's theorem in [2], we did not succeed in proving a corollary to our generalization of Steinmetz's theorem that is analogous to Steinmetz's corollary to his theorem. In the present paper we shall prove a generalization of Steinmetz's theorem which does permit us to prove a corollary analogous to Steinmetz's corollary to his theorem.

Exactly how small Ψ_j must be depends upon a number of considerations; as is evident in [2]. It would *more* than suffice if each $T(r, \Psi_j) = o(T(r, g))$. There, we formulated nice but somewhat involved hypotheses relating the sizes of the Nevanlinna characteristic functions of g and the Ψ_j. Here, facing even greater complexity in determining a valid set of hypotheses if $T(r, \Psi_j) \neq o(T(r, g))$, the wish for simplicity won out: we have hypothesized that each $T(r, \Psi_j) = o(T(r, g))$. Therefore, we do not have a strict generalization of the previous paper [2]. By the nature of the argument it would remain valid if each Ψ_j has a *sufficiently* small positive Nevanlinna characteristic function.

In a yet more recent paper [3], the present authors found all of the solutions $p_i(z)$ in (2) that arise in the original Steinmetz theorem. Obviously a remaining task is to find, when possible, all solutions $p_i(g, \Psi_1, ..., \Psi_n)$ in (2') arising from a generalization of Steinmetz's theorem. The generalization in [3] does not lend itself readily to such a proof. In this paper we first prove what is, aside from differing growth restrictions on the Ψ_j, a yet more general version of Steinmetz's theorem. We shall show later how one can find all solutions arising from this newest version of Steinmetz's theorem, using essentially the construction given in [3]. We now state our latest generalization of Steinmetz's theorem.

Suppose that the $F_j(z)$, for $j = 1, 2, ..., n$, are meromorphic functions, g and the Ψ_j are entire functions, and each Ψ_j has a Nevanlinna characteristic function that is $o(T(r, g))$. Suppose that each $G_j(g, \Psi_1, ..., \Psi_n)$ is a polynomial in the $F_j(g\Psi_j(z))$ for $j = 1, 2, ..., m$.

THEOREM. *If*

$$\sum_j G_j(g, \Psi_1, \Psi_2, ..., \Psi_n)h_j = 0, \tag{3}$$

then there exist polynomials

$$p_j(g, \Psi_1, \Psi_2, ..., \Psi_n)$$

in g *and the*

$$\Psi_1, \Psi_2, ..., \Psi_n$$

that are not all zero such that

$$\sum_j p_j(g, \Psi_1, \Psi_2, ..., \Psi_n)h_j(z) = 0. \tag{4}$$

II. Finding All Related Solutions, and the Corollary

The generalized version of Steinmetz's theorem given in [2] did not permit us to find all solutions related to that generalization of Steinmetz's theorem, since the construction in [3] that produced all solutions involved applying Steinmetz's theorem repeatedly, requiring that the assumed form of the coefficients, $F_j(z)$, be closed under addition, subtraction, and multiplication. The coefficients appearing in [2] were not closed under these operations and this would have made the second and all subsequent applications of the generalized version of Steinmetz's theorem impossible. However, the G_j, above, are closed under addition, subtraction, and multiplication. It follows, therefore, that all solutions arising in the present generalization of Steinmetz's theorem may be found by using the present generalization of Steinmetz's theorem in the argument of [3].

The proof of a generalization of the corollary to Steinmetz's theorem as a corollary of a generalization of Steinmetz's theorem faces a similar difficulty. The argument given by Steinmetz potentially makes use of his theorem several times, so that in a generalization following the same proof but using a generalized Steinmetz theorem, the coefficients must be of a form closed under addition, subtraction, and multiplication. Our present generalization of Steinmetz's theorem in Steinmetz's proof of his corollary then yields:

COROLLARY. *Under the conditions of the theorem, there exist polynomials* $q_j(g, \Psi_1, \Psi_2, ..., \Psi_n)$, *not all zero, such that*

$$\sum_j q_j(g, \Psi_1, \Psi_2, ..., \Psi_n) G_j(g, \Psi_1, \Psi_2, ..., \Psi_n) = 0.$$

III. THE PROOF OF THE THEOREM

Proof of Theorem. As was pointed out in [1] at the end of that paper, the purely algebraic arguments of the proof there would go through even if the functions $F_j(g)$ were replaced by functions $F_j(z, w)$ (evaluated at $w = g$), where at $w = 0$, $F_j(z, w)$ is given by the power series $\sum_i a_{ij} w^i$, the $a_{ij} = a_{ij}(z)$ are each entire functions of z, and every $T(r, a_{ij}(z))$ is $o(T(r, g))$. We used such an argument in [2], where the $F_j(z, w)$'s were our $F_j(g\Psi_j(z))$, for entire functions F_j. It was noted in [2] that each series converged in a disk about each point z_0 where g equaled 0 and in a disk around any point where every $|g(z)\Psi_j(z)|$ was less than some fixed positive real number. This was enough analytically to ensure that an argument patterned after the argument in [1] could be pushed through. The only changes in hypotheses that we are interested in making at this time are that the basic equation replacing Eq. (1′) above must have the form

$$\sum_{l=1}^{m} G_l h_l(z) = 0,$$

where for some natural number n each G_l is a polynomial in

$$F_1(g\Psi_1), ..., F_n(g\Psi_n),$$

where the Ψ_i are entire functions (not necessarily distinct). Here the F_j and the Ψ_i are each to be entire functions. At first the type of argument used in [2] appears to go through. The algebraic part never raises any problem.

The analytic part raises only one problem, with which we shall deal shortly. That section of the analytic part (concerning whether the series represent the functions G_l when $|g(z)\Psi_j(z)|$ is small) comes down to whether the different series represent the $F_j(g\Psi_j(z))$ when $|g(z)\Psi_j(z)|$ is small. This has been answered affirmatively in [2]. Furthermore the condition that the Ψ's each have a Nevanlinna characteristic function that is $o(T(r, g))$ means that too high a growth on the part of the Ψ's can never invalidate the Nevanlinna theory estimates used to prove the generalization of Steinmetz's theorem in [2]. In [2] we needed the hypothesis that the coefficient of h_1, called F_1 there and G_1 here, did not vanish at the origin; hence, it satisfied the requirement that (using the present notation)

$$\left|\frac{1}{G_1}\right|$$

be bounded away from zero when each $|\Psi_i g|$ was small. This was for use in expression (4) of [2]. It would have sufficed there; however, if we had not required that

$$\left|\frac{1}{G_1}\right|$$

be bounded away from zero when each $|\Psi_i g|$ was small but instead had multipled (4) by a factor of

$$q_t(\Psi_1, \dots, \Psi_n)g^t,$$

which is the lowest order term in g in the expansion of G_1 in powers of g, then

$$\left|\frac{q_t(\Psi_1, \dots, \Psi_n)g^t}{G_1}\right|$$

would have been bounded when each $|\Psi_i g|$ was small. Thus, the same asymptotic argument would have sufficed since the Nevanlinna characteristic function of this new factor is at most $(t + o(1))T(r, g)$, which is asymptotically negligible in the argument there. The only difference that could result would be in the degrees to which the Ψ_i's appear in the constructed polynomials, which, since each

$$T(r, \Psi_i) = o(T(r, g)),$$

is irrelevant to the estimates. This proves the theorem.

References

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